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Exact Results for a Random Frustrated Ising Model on the Kagomé Lattice

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We perform a slight modification of the decoration–decimation transformation which allows us to map the homogeneous Ising model on the honeycomb lattice on an inhomogeneous Ising model on the Kagomé lattice. Then, we obtain exact results for a class of random bond Ising model on the Kagomé lattice with competing interactions and show that the different types of frustration make the critical point of the pure model disappear.

KEY WORDS: Decoration; inhomogeneous Ising model; disorder; frustration; exactly solvable model.

1. INTRODUCTION

The study of random systems (amorphous or glassy systems, spin glasses,[1] dilute alloys,[2] etc.) is one of the most important topics in condensed matter physics. From the theoretical side, the study of disordered systems by means of random bond Ising models has been the subject of intensive research in recent years. In the most popular version of this type of model, the interactions take randomly one of two possible values at each bond of the lattice. To study the dilution that arises when nonmagnetic atoms are diluted in a magnetic matrix, the two possible values are J and zero. Another possibility is the random bond $\pm J$ Ising model (frustration model), which contains the essential effects of disorder and of competition between the interactions needed to describe spin-glass systems. Among these systems, spin glasses have the least understood equilibrium properties; this lack of understanding originates from the formidable difficulties

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encountered in the analytical approaches when quenched averages are performed.

Since in this paper we will be mainly concerned with exact results obtained for models with competing interactions, we will briefly review previous work in this direction. The first step was, naturally, to look for exact solutions of one-dimensional models. However, a chain of spins with random ferromagnetic and antiferromagnetic bonds is not frustrated, because a single change of spin variables makes the problem trivial. Thus, it is necessary to include a magnetic field in order to obtain a model exhibiting frustration effects. This class of models was studied by several authors,[3-6] but the analytical results obtained so far are limited to the low-temperature behavior. For example, Gardner and Derrida[5] derived a formula for the zero-temperature magnetization in a weak magnetic field and for arbitrary bond distribution, which was previously suggested by Chen and Ma.[6]

The one-dimensional Ising model in a random field also has been considered by many authors.[7-11] The particular case of a binary random field ($\pm h$) is equivalent to the ($\pm J$) random bond Ising chain in a uniform field. Rather than reviewing the increasingly important work on this model, we merely cite a recent paper[12] in which an Ising chain in a varying magnetic field is solved. In this paper, the free energy and its derivatives with respect to temperature are exactly computed at a particular fixed value of the temperature. This limitation also arises in our procedure and is a characteristic of a large class of analytical approaches, such as the disorder solutions.[13,14]

If there are few exact results for one-dimensional disordered models, the situation is even worse in two dimensions. In fact, there are no exact results with full two-dimensional randomness for short-range interactions. The models exactly solved have "striped randomness," such as the dilute model proposed by McCoy[15] and the frustrated model analyzed by Wolff and Zittartz.[16] In this type of layered model, only the bonds in one direction are random variables.

In many dimensions, one could consider the work of Nishimori,[14] who obtained disorder solutions for the short-range frustrated Ising model. Although he gave useful information about the phase diagram of this model, he was unable to extract some singularity and thus to study the profound effects of frustration.

On the other hand, much analytical work has been devoted to infinite-range interaction models, in particular, that of Sherrington and Kirkpatrick.[17,18] However, the infinite-range models may be considered as infinite-dimensional ones, which cannot reflect the dependence of various properties of spin glasses on dimension.

In this paper, we present exact results for a disordered Ising model on a Kagomé lattice (KIM) at a fixed temperature. In fact, we are able to construct a class of inhomogeneous Ising models on a Kagomé lattice for which the free energy can be exactly computed. This construction involves the use of the decoration–decimation procedure[19] modified in such a way that a bond K of the original lattice is replaced by a series array of two parameters L and M, which can be chosen differently at each bond of this lattice. After decimation, the coupling constants may be different at each bond of the new lattice. Then, one can perform a mapping of the homogeneous Ising model on the honeycomb lattice (HIM) to an inhomogeneous KIM. This procedure is explained in detail in Section 2.

In Section 3 we solve as a particular case a disordered model obtained by taking the intermediate variables L and M as random variables with a two-delta probability distribution. We consider some possible sets of values that can take the random variables, and work out the free energy. In spite of the simplicity of the distribution adopted, we can obtain a model with competing interactions and with different types of frustration. We show that in all cases the effect of frustration consists in eliminating the critical point of the pure KIM.

In Section 4 we briefly present, as another application of our transformation, the exact results obtained for a nonrandom, quasiperiodic model.

Finally, in Section 5, we summarize our results, discuss our main conclusions, and indicate new possible applications and further developments of the transformation considered in this paper.

2. NONUNIFORM DECORATION TRANSFORMATION

Let us consider the anisotropic Ising model on a honeycomb lattice (HIM) with a Hamiltonian given by

$$H = -\sum_{(ij)} \{s_{ij}(K^{(1)}t_{ij} + K^{(2)}t_{ij+1} + K^{(3)}t_{i-1j})\} \tag{2.1}$$

where $\{s_{ij}\}$ and $\{t_{ij}\}$ belong to two different sublattices, indicated by solid and open circles, respectively, in Fig. 1, and where the sum extends over the sites of one of these sublattices, for example, the one of circles. Notice that each of these sublattices is in fact a square lattice. This, perhaps unusual, labeling was found to be very convenient for performing the following transformation. The interactions $K^{(\alpha)}$ ($\alpha = 1, 2, 3$) include the factor $1/k_B T$, where k_B is the Boltzmann constant and T the temperature.

The partition function for this model is defined by

$$Z_h(\{K^{(\alpha)}\}) = \sum_{s,t} \exp(-H) \tag{2.2}$$

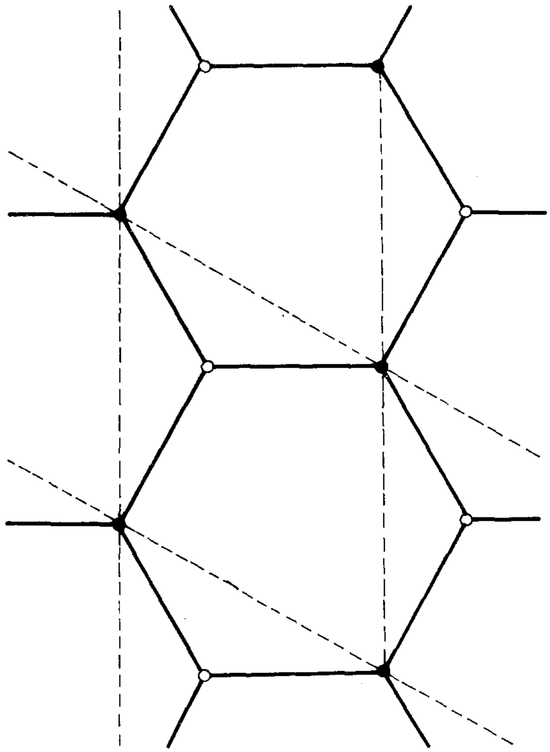

Fig. 1. The two sublattices (●) $\{s_{ij}\}$ and (○)$\{t_{ij}\}$ that build up the honeycomb lattice.

where the sum extends over all the spin configurations of the sites s_{ij} and t_{ij}.

Now we perform the decoration of the honeycomb lattice, which consists in introducing an extra spin variable $\sigma^{(\alpha)}$ on the middle point of every bond of the original lattice. In the usual procedure[19] one replaces each bond $K^{(\alpha)}$ by two equal bonds $L^{(\alpha)}$ as shown in Fig. 2.

Then, in terms of the new variables $L^{(\alpha)}$, the partition function (2.2) may be rewritten as

$$Z_h = (A^{(1)}A^{(2)}A^{(3)})^{N_h/2} \sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} \sum_{s,t} \exp\left\{\sum_{(ij)} [L^{(1)}\sigma_{ij}^{(1)}(s_{ij}+t_{ij}) + L^{(2)}\sigma_{ij}^{(2)}(s_{ij}+t_{ij+1}) + L^{(3)}\sigma_{ij}^{(3)}(s_{ij}+t_{i-1j})]\right\} \tag{2.3}$$

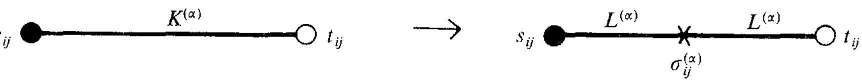

Fig. 2. The uniform decoration transformation.

where

$$\tanh^2(L^{(\alpha)}) = \tanh(K^{(\alpha)}) \tag{2.4a}$$

$$A^{(\alpha)} = \tfrac{1}{2}[\cosh(2L^{(\alpha)})]^{-1/2}, \qquad \alpha = 1, 2, 3 \tag{2.4b}$$

and N_h is the number of sites of the honeycomb lattice.

Now we propose a generalization of this procedure by replacing each bond $K^{(\alpha)}$ by two different bonds $L^{(\alpha)}$ and $M^{(\alpha)}$. This allows us to perform a nonuniform decoration by taking different $L^{(\alpha)}$ (and correspondingly $M^{(\alpha)}$) for each bond of the lattice. For convenience we denote by $L_{ij}^{(\alpha)}$ ($M_{ij}^{(\alpha)}$) the interactions with one end at site s_{ij} (t_{ij}), as shown in Fig. 3.

We can verify that we still have only one constraint per bond, since (2.4a) is now replaced by

$$\begin{aligned}
\tanh(L_{ij}^{(1)}) \tanh(M_{ij}^{(1)}) &= \tanh(K^{(1)}) \\
\tanh(L_{ij}^{(2)}) \tanh(M_{ij+1}^{(2)}) &= \tanh(K^{(2)}) \\
\tanh(L_{ij}^{(3)}) \tanh(M_{i-1j}^{(3)}) &= \tanh(K^{(3)})
\end{aligned} \tag{2.5}$$

Therefore, we have, instead of (2.2),

$$\begin{aligned}
Z_h = \prod_{(ij)} [A_{ij}^{(1)} A_{ij}^{(2)} A_{ij}^{(3)}] \sum_{\sigma^{(1)}\sigma^{(2)}\sigma^{(3)}} \sum_{st} \exp \Big\{ \sum_{(ij)} [s_{ij}(L_{ij}^{(1)} \sigma_{ij}^{(1)} \\
+ L_{ij}^{(2)} \sigma_{ij}^{(2)} + L_{ij}^{(3)} \sigma_{ij}^{(3)}) + M_{ij}^{(1)} t_{ij} \sigma_{ij}^{(1)} + M_{ij+1}^{(2)} t_{ij+1} \sigma_{ij}^{(2)} \\
+ M_{i-1j}^{(3)} t_{i-1j} \sigma_{ij}^{(3)}] \Big\}
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
A_{ij}^{(1)} &= \tfrac{1}{2}[\cosh(L_{ij}^{(1)} + M_{ij}^{(1)}) \cosh(L_{ij}^{(1)} - M_{ij}^{(1)})]^{-1/2} \\
A_{ij}^{(2)} &= \tfrac{1}{2}[\cosh(L_{ij}^{(2)} + M_{ij+1}^{(2)}) \cosh(L_{ij}^{(2)} - M_{ij+1}^{(2)})]^{-1/2} \\
A_{ij}^{(3)} &= \tfrac{1}{2}[\cosh(L_{ij}^{(3)} + M_{i-1j}^{(3)}) \cosh(L_{ij}^{(3)} - M_{i-1j}^{(3)})]^{-1/2}
\end{aligned} \tag{2.7}$$

Fig. 3. The modified, nonuniform, decoration transformation.

As usual, after the decoration step, the old spins $\{s_{ij}\}$ and $\{t_{ij}\}$ become decoupled and can be explicitly summed out. This decimation transforms the honeycomb lattice into the Kagomé lattice (see Fig. 4).

Then, one can write

$$Z_h(\{K^{(\alpha)}\}) = \prod_{(ij)} [A_{ij}^{(1)} A_{ij}^{(2)} A_{ij}^{(3)} B_{ij} C_{ij}]\, Z_{\mathrm{Kag}}(\{\bar{P}, \bar{R}\}) \tag{2.8a}$$

where

$$\begin{aligned} Z_{\mathrm{Kag}}(\{\bar{P}, \bar{R}\}) = \sum_{\sigma^{(1)}\sigma^{(2)}\sigma^{(3)}} \exp \Big\{ \sum_{(ij)} [& P_{ij}^{(1)} \sigma_{ij}^{(1)} \sigma_{ij}^{(2)} \\ & + P_{ij}^{(2)} \sigma_{ij}^{(1)} \sigma_{ij}^{(3)} + P_{ij}^{(3)} \sigma_{ij}^{(2)} \sigma_{ij}^{(3)} + R_{ij}^{(1)} \sigma_{ij-1}^{(1)} \sigma_{i+1j}^{(3)} \\ & + R_{ij}^{(2)} \sigma_{ij}^{(2)} \sigma_{i+1j}^{(3)} + R_{ij}^{(3)} \sigma_{ij}^{(2)} \sigma_{ij-1}^{(1)}] \Big\} \end{aligned} \tag{2.8b}$$

Using the well known star–triangle relations (Fig. 4b), we have that

$$\tanh(2P_{ij}^{(1)}) = \frac{\sinh(2L_{ij}^{(2)}) \sinh(2L_{ij}^{(3)})}{\cosh(2L_{ij}^{(1)}) + \cosh(2L_{ij}^{(2)}) \cosh(2L_{ij}^{(3)})} \tag{2.9a}$$

$$\tanh(2R_{ij}^{(1)}) = \frac{\sinh(2M_{ij}^{(2)}) \sinh(2M_{ij}^{(3)})}{\cosh(2M_{ij}^{(1)}) + \cosh(2M_{ij}^{(2)}) \cosh(2M_{ij}^{(3)})} \tag{2.9b}$$

and the remaining coupling constants are calculated by similar expressions obtained by cyclic permutations of the indices (1, 2, 3). The star–triangle relations also establish that

$$B_{ij} = \left\{ 4 \left[\sum_{\alpha=1}^{3} \sinh^2(2L_{ij}^{(\alpha)}) + 2 \prod_{\alpha=1}^{3} \cosh(2L_{ij}^{(\alpha)}) + 2 \right] \right\}^{1/4} \tag{2.10a}$$

$$C_{ij} = \left\{ 4 \left[\sum_{\alpha=1}^{3} \sinh^2(2M_{ij}^{(\alpha)}) + 2 \prod_{\alpha=1}^{3} \cosh(2M_{ij}^{(\alpha)}) + 2 \right] \right\}^{1/4} \tag{2.10b}$$

Finally, by defining the free energies per spin in the usual way as

$$f_h = -\frac{1}{N_h} \ln Z_h, \qquad f_{\mathrm{Kag}} = -\frac{1}{N_{\mathrm{Kag}}} \ln Z_{\mathrm{Kag}}$$

where $N_{\mathrm{Kag}} = 3N_h/2$, and taking into account (2.8a), we obtain

$$f_{\mathrm{Kag}} = \frac{1}{N_{\mathrm{Kag}}} \sum_{ij} [\ln A_{ij}^{(1)} + \ln A_{ij}^{(2)} + \ln A_{ij}^{(3)} + \ln B_{ij} + \ln C_{ij}] + \frac{2}{3} f_h \tag{2.11}$$

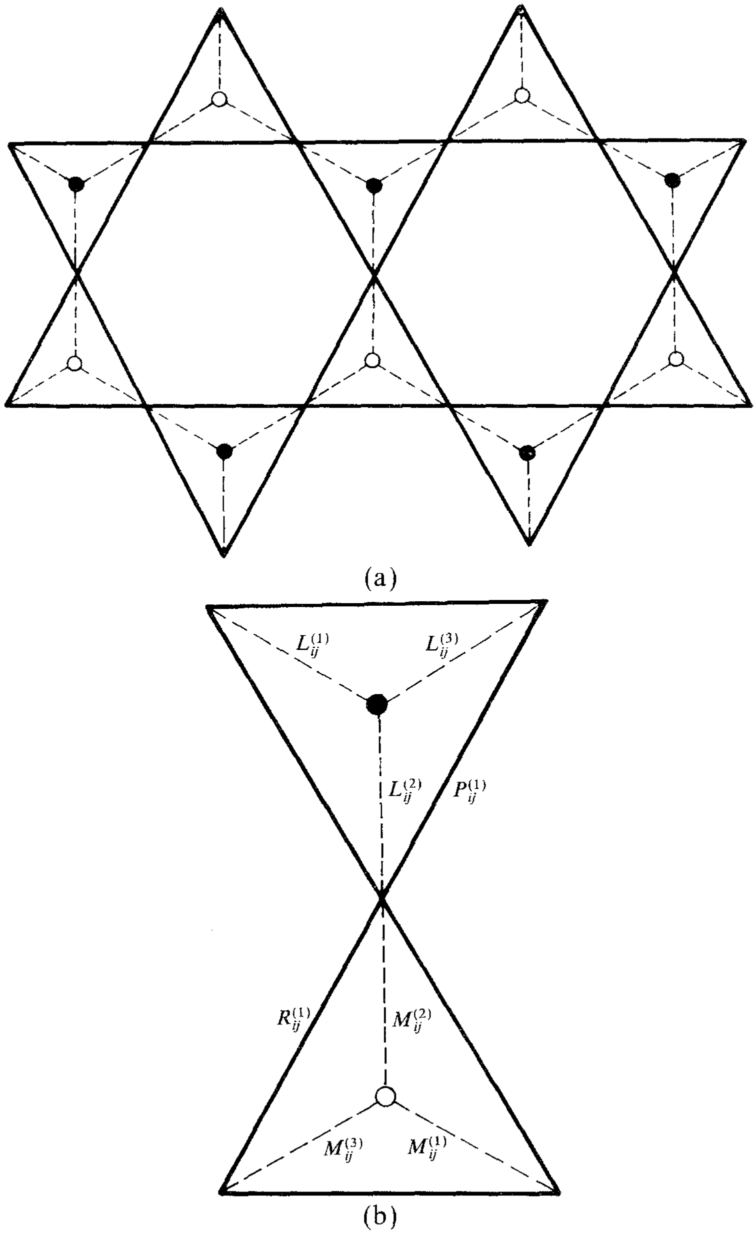

Fig. 4. (a) (—) The Kagomé lattice resulting after the decimation of the sites (●, ○) of (- -) the honeycomb lattice. (b) An elementary cell of the Kagomé lattice composed of a "P-triangle" (upper triangle) and an "R-triangle" (lower). The sites of the honeycomb lattice and the intermediate interactions $L_{ij}^{(\alpha)}$ and $M_{ij}^{(\alpha)}$ arising after the decimation step are also shown.

with[19]

$$f_h = \frac{-1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \left\{ \frac{1}{2} [1 + C^{(1)} C^{(2)} C^{(3)} - S^{(1)} S^{(3)} \cos(\omega_1 - \omega_2) - S^{(1)} S^{(2)} \cos \omega_1 - S^{(2)} S^{(3)} \cos \omega_2] \right\} d\omega_1 \, d\omega_2 \tag{2.12}$$

where $C^{(\alpha)} \equiv \cosh(2K^{(\alpha)})$ and $S^{(\alpha)} \equiv \sinh(2K^{(\alpha)})$.

In this way, we have expressed the free energy of the nonuniform KIM in terms of the free energy of the uniform HIM.

The relations (2.5) and (2.9) can be used to choose the parameters $L_{ij}^{(\alpha)}$ as the independent variables that govern the inhomogeneity of the KIM. It is easy to verify that the number of independent parameters $L_{ij}^{(\alpha)}$ is equal to the number of sites of the Kagomé lattice (N_{Kag}) and equal to the half of the interactions $P_{ij}^{(\alpha)}$ and $R_{ij}^{(\alpha)}$.

If the system is a disordered one, that is, in our case, a random bond Ising model, the expression (2.11) is valid for a particular realization of $\{L_{ij}^{(\alpha)}\}$. The quenched free energy, after taking the average over the disorder, is given by the following expression:

$$\bar{f}_{\text{Kag}} = \prod_{ij} \iiint dL_{ij}^{(1)} \, dL_{ij}^{(2)} \, dL_{ij}^{(3)} \, \rho(L_{ij}^{(1)} \, \rho(L_{ij}^{(2)}) \, \rho(L_{ij}^{(3)}) \, f_{\text{Kag}} \tag{2.13}$$

where $\rho(L_{ij}^{(\alpha)})$ is the probability distribution of $L_{ij}^{(\alpha)}$.

As was stated above, using (2.5) and (2.9b) (and its cyclic permutations), one can express the variables $R_{ij}^{(\alpha)}$ in terms of the parameters $L_{ij}^{(\alpha)}$. Then, taking into account the relation (2.9a) and its cyclic permutations, one can see that the interactions $R_{ij}^{(\alpha)}$ are related to the $P_{ij}^{(\alpha)}$ through temperature-dependent equations. This implies that the temperature is fixed. This important limitation of our transformation also arises in other procedures employed to obtain exact results in nonuniform models (see, for example, Refs. 12, 20, and 21).

Finally, remark that the parameters $L_{ij}^{(\alpha)}$, $M_{ij}^{(\alpha)}$, and $K^{(\alpha)}$ are not restricted to real values. Only the variables $P_{ij}^{(\alpha)}$ and $R_{ij}^{(\alpha)}$, which are the interactions of the KIM, must be real. In the next section, we will take advantage of this freedom by choosing $L_{ij}^{(\alpha)}$ or $K^{(\alpha)}$ in such a way as to introduce frustration in the nonuniform KIM.

3. RANDOM BOND ISING MODEL ON THE KAGOMÉ LATTICE

In this section we apply the formalism described above to a class of random bond Ising models obtained by assuming that the $L_{ij}^{(\alpha)}$ are independent random variables with a two-delta function probability distribution $(0 \leqslant p \leqslant 1)$:

$$\rho(L_{ij}^{(\alpha)}) = p\delta(L_{ij}^{(\alpha)} - L_1) + (1-p)\,\delta(L_{ij}^{(\alpha)} - L_2) \tag{3.1}$$

where L_1 and L_2 can take, in general, complex values. Perhaps it would make more physical sense to give from the beginning the probability distributions of the interactions $P_{ij}^{(\alpha)}$ considered as the independent variables, i.e., inverting (2.9a), but the calculations would become somewhat more complicated.

From (2.9a) and (2.9b), and taking for simplicity the isotropic case $K^{(\alpha)} = K$, it can be seen that each of the variables $P_{ij}^{(\alpha)}$ and $R_{ij}^{(\alpha)}$ can take six possible values $P_1,..., P_6$ and $R_1,..., R_6$. These values, together with their respective probabilities, are shown in Appendix A. One must keep in mind that the effective probability of having a given value of the interactions, or an interaction of a given sign, is not p, but a function, which can be a constant, of p.

The quenched free energy of this model can be computed exactly from (2.13) after replacing the probability distributions given by (3.1) and taking into account (2.7), (2.10), and (2.11). This calculation is greatly simplified, since we have an equal contribution from all the sites (ij) of the underlying square lattice [number of sites $(ij) = N_{\text{Kag}}/3$]. Moreover, the integrals, due to the delta functions involved in the probability distributions, reduce to a sum over the possible values of $\ln A_{ij}^{(\alpha)}$, $\ln B_{ij}$, $\ln C_{ij}$, each term of this sum been weighted by the probabilities $a_m(p)$, $b_m(p)$, and $c_m(p)$, respectively. Thus, one can write

$$\overline{f_{\text{Kag}}} = -\frac{1}{3}\sum_m [3a_m(p)\ln A_m + b_m(p)\ln B_m + c_m(p)\ln C_m] + \frac{2}{3}f_h(K) \tag{3.2}$$

where A_m, B_m, and C_m are functions of L_1, L_2, and K. The final expression for the free energy for general values of L_1, L_2, and K is given in Appendix B.

Now we study the behavior of our model in the space of complex parameters (L_1, L_2, K).

Consider first the case $L_1 = -L_2$, which is the simplest nontrivial case that leads to competing interactions. From the expression given in Appendix A, we see that P_j and R_j $(j = 1,..., 6)$ can take the following values:

$$P_j = \pm\frac{1}{4}\ln\frac{1 + 3\tanh^2 L_1}{1 - \tanh^2 L_1} \tag{3.3a}$$

$$R_j = \pm\frac{1}{4}\ln\frac{\tanh^2 L_1 + 3\tanh^2 K}{\tanh^2 L_1 - \tanh^2 K} \tag{3.3b}$$

For this particular case, the final model reduces to an ordered (nonrandom) one. That is, the free energy, after taking $t_1=-t_2$ ($t_\mu=\tanh L_\mu$ and $t_K=\tanh K$ in (B.1) and using (2.11) with $K^{(\alpha)}=K$ becomes

$$f_{\text{Kag}}=-\frac{1}{3}\left[-2\ln t_1-\frac{3}{2}\ln(1-t_K^2)+\frac{3}{4}\ln(1-t_1^2)\right.$$
$$\left.+\frac{3}{4}\ln(t_1^2-t_k^2)+\frac{1}{4}\ln(1+3t_1^2)+\frac{1}{4}\ln(t_1^2+3t_k^2)-\ln 2\right]+\frac{2}{3}f_h \qquad (3.4)$$

where

$$f_h=-\frac{1}{24\pi^2}\int_0^{2\pi}\int_0^{2\pi} dw_1\, dw_2 \ln\left(\frac{1}{2}\{1+\cosh^3(2K)\right.$$
$$\left.-\sinh^2(2K)[\cos(\omega_1-\omega_2)+\cos\omega_1+\cos\omega_2]\}\right) \qquad (3.5)$$

i.e., independent of p. One can verify that (3.4) is also valid for the case $L_1=L_2$. This result may be understood from the fact that by performing a change of sign of some of the spin variables, one can always change the signs of the interactions in such a way as to obtain an ordered model. We emphasize that even this ordered model, which for certain values of the parameters can be frustrated, is not contained in the exact solution of the anisotropic KIM. This result allows us to separate the effect of randomness from other effects, which originate from dilution or frustration.

We now discuss our model in the following subspaces of parameters: (I) L_1, L_2, and K are real values; (II) L_1 and L_2 are real and $K=ik$, $i=\sqrt{-1}$, k real; (III) $L_\mu=il_\mu$, l_μ real ($\mu=1, 2$), and K real.

Subspace I. For this case, the relations (2.5) lead to the restriction

$$|L_\mu|>|K|, \qquad \mu=1,2 \qquad (3.6)$$

which implies that L_μ cannot vanish ($K=0$ is obviously trivial). Then, from the expressions for P_j and R_j in Appendix A, it can be seen that these parameters are also different from zero. Thus, the dilute model cannot be studied within this subspace.

The other interesting nonuniform model is that of competing interactions. From (A.1) we see that in order to obtain interactions with opposite signs in this subspace, one must take sign $L_1\neq$ sign L_2. Then, the interactions P_j and R_j ($j=1,\ldots,6$) take the signs $(+,+,-,-,+,+)$ and $(+,+,-,-,+,+)$, respectively (in the following we will refer to this type of sign set as a "sign rule"). From this sign rule one obtains that the effective probability $\tilde{p}$ for the interactions to be positive is given by

$$\tilde{p} = 1 - 2p + 2p^2 \tag{3.7}$$

and, since $0 \leqslant p \leqslant 1$, this probability is restricted to $1/2 \leqslant \tilde{p} \leqslant 1$. It is important to notice that, in spite of the presence of competing interactions, the resulting Kagomé lattice is unfrustrated, i.e., it has no frustrated closed paths (as usual, we call a given path on the lattice "frustrated" if the product of the signs of the interactions along that path is equal to -1). The ultimate reason for the absence of frustration is given by the relations (2.5) and (2.9), which are implicit in the sign rule.

The analysis of the case $L_1 = -L_2$ is almost trivial, as expected from the comment made inmediately below (3.4). In effect, in this particular case, it is easy to see that a single change of signs of some of the spin variables is sufficient to make all the interactions positive.

As can be seen from (B.1), due to the analyticity of the transformation, there are no singularities in the terms between brackets, except for the trivial one at $K = +\infty$. Then, the absence of frustration manifests itself in that the critical behavior of the random bond KIM is the same of that of the pure HIM that arises as a singularity of the integral (3.5) when $\cosh(2K_c) = 2$.

Subspace II. After replacing $K = ik$ in the condition (2.5) it can be seen that $M_{ij}^{(\alpha)}$ must be imaginary numbers, i.e., $M_{ij}^{(\alpha)} = im_{ij}^{(\alpha)}$, with $m_{ij}^{(\alpha)}$ real. Then, this condition may be rewritten as

$$\tanh L_\mu \operatorname{tg} m_\mu = \operatorname{tg} k, \qquad \mu = 1, 2 \tag{3.8}$$

In order to obtain physical interactions, we can see from the expressions given Appendix A that the condition

$$|\operatorname{tg} k| < \frac{1}{\sqrt{3}} \min\{|\tanh L_1|, |\tanh L_2|\} \leqslant \frac{1}{\sqrt{3}} \tag{3.9}$$

must be satisfied.

At first sight, condition (3.8) seems to allow us to study a dilute random model by taking $t_1 \to 0$ and simultaneously tg $(m_1 \to \infty$ so as to keep k constant and t_2 nonzero. However, from (3.9) we conclude that $k \to 0$, thus reducing the model to a trivial one.

We now turn to the study of frustration. Note that for both cases sign $L_1 =$ sign L_2 and sign $L_1 \neq$ sign L_2 we obtain competing interactions P_j and R_j with the sign rules $(+,..., +)$ and $(-,..., -)$ for the former case and $(+, +, -, -, +, +)$ and $(-, -, +, +, -, -)$ for the latter. From these sign rules it follows immediately that for both cases the effective probability for having a positive interaction is $\tilde{p} = 1/2$, independent of p.

For both situations, both the R-triangles and the hexagons can hold frustration, while the P-triangles cannot (see Fig. 5). This type of frustration eliminates the critical point of the pure KIM. This very important effect can be seen from (3.2), since the sum gives an analytical contribution, while f_h given by (3.5) can be rewritten as

$$f_h = -\frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} dw_1\, dw_2 \ln \Big(\frac{1}{2} \{ 1 + \cos^3(2k) + \sin^2(2k)[\cos(\omega_1 - \omega_2) + \cos w_1 + \cos \omega_2] \} \Big) \tag{3.10}$$

A careful analysis of (3.10) shows that there are only two singular points. These points, located at $k = \pi/4$ and $\pi/2$, correspond to $\operatorname{tg} k = 1$ and $+\infty$ and therefore they are excluded due to the condition (3.9).

Subspace III. Again (2.5) implies that $M_{ij}^{(\alpha)}$ must be imaginary numbers. In terms of real values this condition takes the form

$$\operatorname{tg} l_\mu \operatorname{tg} m_\mu = -\tanh K, \qquad \mu = 1, 2 \tag{3.11}$$

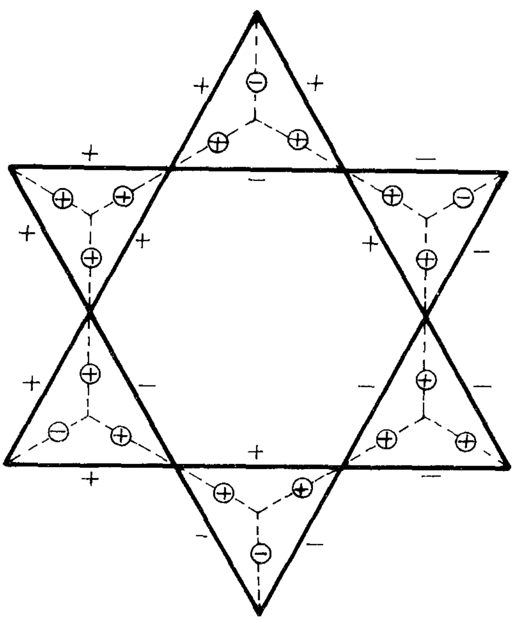

Fig. 5. In subspace II of parameters, a possible set of signs of the random interactions $L_{ij}^{(\alpha)}$ and $M_{ij}^{(\alpha)}$ (dashed lines) that leads to frustrated hexagons and R-triangles and nonfrustrated P-triangles, according to the sign rules of this subspace.

Another set of conditions arises by imposing that the interactions P_j and R_j given in Appendix A take physical, i.e., real, values. Taking into account the symmetry of those expressions with respect to l_1 and l_2, one can show that the necessary and sufficient conditions to be satisfied by l_1, l_2, and K are

$$|\text{tg}\, l_\mu| < \frac{1}{\sqrt{3}}, \qquad \mu = 1, 2 \tag{3.12a}$$

$$|\tanh K| < \frac{1}{\sqrt{3}} \min\{|\text{tg}\, l_1|, |\text{tg}\, l_2|\} \tag{3.12b}$$

Of course, (3.12a) and (3.12b) imply that

$$|\tanh K| < \frac{1}{3} \tag{3.13}$$

It is easy to verify that, as in the preceding subspaces, the study of the dilute model could only be done at the trivial point $K = 0$.

In order to determine the presence of frustration we observe from expressions (A.1) that if sign l_1 = sign l_2 all the interactions P_j and R_j become negative, and if sign $l_1 \neq$ sign l_2 one has for both types of interactions the sign rule $(-, -, +, +, -, -,)$. Then, for both cases all the triangles are always frustrated, while the hexagons are unfrustrated. The difference between these two situations is that the first one (sign l_1 = sign l_2) is in fact a disordered antiferromagnetic model with an effective probability $\tilde{p} = 0$, and the second (sign $l_1 \neq$ sign l_2) is a model with true competing interactions with

$$\tilde{p} = 2p(1-p) \tag{3.14}$$

which implies $0 \leqslant \tilde{p} \leqslant 1/2$.

As in the preceding subspace, the frustration eliminates the critical point of the pure model, since the singularity of the integral (3.5) located at $\cosh(2K_c) = 2$ leads to $\tanh(K_c) = 1|\sqrt{3}$, which is excluded by (3.13).

4. APPLICATION TO A QUASIPERIODIC MODEL

In this section we apply the procedure developed in Section 2 to a deterministic inhomogeneous model that is possibly interesting for the study of incommensurate systems. A model of this type is obtained by imposing a quasiperiodic variation on the independent parameters $L_{ij}^{(\alpha)}$; for example,

$$L_{ij}^{(\alpha)} = \lambda^{(\alpha)} + \delta^{(\alpha)} \cos(2\pi iw) \cos(2\pi jw) \tag{4.1}$$

where w is an irrational number. If we consider subspace I of parameters defined in the preceding section, that is, $L_{ij}^{(\alpha)}$ and K real numbers, the restriction $|L_{ij}^{(\alpha)}| > |K|$ implies that

$$|\lambda^{(\alpha)}| - |\delta^{(\alpha)}| > |K| \tag{4.2}$$

After replacing (4.1) in (2.5), (2.7), (2.9), and (2.10), it follows that the variables $P_{ij}^{(\alpha)}$ and $R_{ij}^{(\alpha)}$ on one hand and $A_{ij}^{(\alpha)}$, B_{ij}, and C_{ij} on the other also constitute quasiperiodic sequences. In fact, we can write

$$\begin{aligned} A_{ij}^{(\alpha)} &= A^{(\alpha)}(u_i, u_j) \\ B_{ij} &= B(u_i, u_j) \\ C_{ij} &= C(u_i, u_j) \end{aligned} \tag{4.3}$$

where $u_i = iw$ and $u_j = jw$ are two equidistributed sequences (mod 1). In this section, the variable i should not be confused with $\sqrt{-1}$.

In order make the notation more compact, let us call

$$\Phi(u_i, u_j) = \ln[A_{ij}^{(1)} A_{ij}^{(2)} A_{ij}^{(3)} B_{ij} C_{ij}] \tag{4.4}$$

Then we can rewrite (2.11) in the thermodynamic limit $N_{\text{Kag}} \to \infty$ as

$$\begin{aligned} f_{\text{Kag}} &= \lim_{N_{\text{Kag}} \to \infty} \frac{1}{N_{\text{Kag}}} \sum_{ij} \Phi(u_i, u_j) + \frac{2}{3} f_h \\ &= \lim_{L_x, L_y \to \infty} \frac{1}{L_x L_y} \sum_{i=1}^{L_x} \sum_{j=1}^{L_y} \Phi(u_i, u_j) + \frac{2}{3} f_h \end{aligned} \tag{4.5}$$

where obviously $N_{\text{Kag}} = L_x L_y$. Now applying a straightforward generalization of a lemma due to Weyl,[12] we obtain the exact free energy of the quasiperiodic model:

$$f_{\text{Kag}} = \int_0^1 dx \int_0^1 dy\, \Phi(x, y) + \frac{2}{3} f_h \tag{4.6}$$

Of course, the integrand of the last equation is a very complicated function of the variables x and y, but it may be computed numerically with the desired accuracy.

We think that it would be more interesting to analyse the quasiperiodic models obtained by considering subspaces II and III, that is, taking $L_{ij}^{(\alpha)}$ or $K^{(\alpha)}$ as imaginary numbers. These models would have different types of nonuniform frustration and, in consequence, deserve a more detailed study.

5. CONCLUSIONS

We generalized the decoration transformation by replacing a bond $K^{(\alpha)}$ of the original lattice by two interactions $L_{ij}^{(\alpha)}$ and $M_{ij}^{(\alpha)}$, which may be taken differently at each bond of the lattice. This provides a new degree of freedom, which can be used to transform a uniform model into a nonuniform one. The price we have to pay for this is that the resulting model is at a fixed temperature.

We applied this transformation to a uniform Ising model on a honeycomb lattice to obtain a nonuniform Ising model on a Kagomé lattice. In particular we considered a disordered KIM with competing interactions. We were also able to study separately the effects of disorder from other features of the ordered model, such as frustration.

In fact, we have seen that the effects of disorder are almost irrelevant, since they do not modify qualitatively, i.e., with respect to criticality, the behavior of the quenched free energy. Of course, since disorder is marginal in the $2d$ Ising model (Harris criterion[22]), other disordered $2d$ Ising models (for example, the "striped randomness" model studied by McCoy[15]) do have critical behaviors different from that of the pure case.

On the other hand, for all the models (ordered or not) where the competing interactions lead to frustration, the critical point of the pure (ferromagnetic) KIM disappears. This is in accord with the now widely accepted view that the spin glass does not exists as a genuine equilibrium phase transition in two dimensions.

In particular, for the subspace III, where the triangles, but not the hexagons, are frustrated, the free energy is essentially the same of that of the pure model, but it cannot become singular due to restrictions on the parameter K. In subspace II, hexagons can also become frustrated and this leads to a profound change in the free energy, which does not merely consist in a limitation on the values of the parameter K. For this subspace the effective probability $\tilde{p}$ is fixed at 1/2, which certainly indicates the strongest competition and is the most commonly studied from the numerical point of view.

For the particular case $L_1 = -L_2$, the possible values of the interactions reduce to two, P and R, which are functions of the parameters L and K. For this case, the temperature is no longer fixed and one can compute thermodynamic properties, such as the specific heat.

The decoration transformation can be generalized to include magnetic fields.[19] This generalization is used to relate the magnetization between two models, for example, the honeycomb and the Kagomé lattice Ising models. In principle, it is possible to reformulate this extended transformation in a similar fashion to that performed in this paper for the transfor-

Table I

$\{L_{ij}^{(\alpha)}\}$	Label of P_j and R_j	Probability
$L_1L_1L_1$	1	p^3
$L_1L_1L_2$	2	$p^2(1-p)$
$L_1L_2L_1$ or $L_2L_1L_1$	3	$2p^2(1-p)$
$L_2L_1L_2$ or $L_1L_2L_2$	4	$2p(1-p)^2$
$L_2L_2L_1$	5	$p(1-p)^2$
$L_2L_2L_2$	6	$(1-p)^3$

mation without magnetic fields. At first sight, the condition of uniform field in the original (honeycomb) model would lead to further correlations between the parameters $L_{ij}^{(\alpha)}$ and $M_{ij}^{(\alpha)}$, but surely sufficient freedom would remain to introduce nonuniformity and so also disorder. Then one could compute the magnetization of our disordered KIM at a fixed temperature.

It also would be worthwhile to obtain information about the susceptibilities of the nonuniform models considered in this paper, by using the already known behavior of the pure honeycomb lattice Ising model, which results from series expansions and other approximate methods. These quantities would allow us a better comprehension of the phases of these nonuniform models.

APPENDIX A

In this Appendix we show the possible values P_j and R_j ($j=1,\dots,6$) that can take the $P_{ij}^{(\alpha)}$ and $R_{ij}^{(\alpha)}$, respectively, in terms of L_1, L_2, and K. These values are obtained from (2.5) and (2.9) after replacing $L_{ij}^{(\alpha)}$ by L_1 or L_2 (see Table I). Taking into account the double-delta probability distribution (3.1), we calculate the probability of each value P_j and R_j:

$$\begin{aligned}
P_1 &= \frac{1}{4}\ln\frac{1+3t_1^2}{1-t_1^2}\\
P_2 &= \frac{1}{4}\ln\frac{(1+t_1^2)^2-4t_1t_2}{(1-t_1^2)^2}\\
P_3 &= \frac{1}{4}\ln\frac{1+t_1^2+2t_1t_2}{1+t_1^2-2t_1t_2}\\
P_4 &= \frac{1}{4}\ln\frac{1+t_2^2+2t_1t_2}{1+t_2^2-2t_1t_2}\\
P_5 &= \frac{1}{4}\ln\frac{(1+t_2^2)^2-4t_1t_2}{(1-t_2^2)^2}\\
P_6 &= \frac{1}{4}\ln\frac{1+3t_2^2}{1-t_2^2}
\end{aligned} \tag{A.1a}$$

$$
\begin{aligned}
R_1 &= \frac{1}{4} \ln \frac{t_1^2 + 3t_K^2}{t_1^2 - t_K^2} \\
R_2 &= \frac{1}{4} \ln \frac{t_2^2(t_1^2 + t_K^2)^2 - 4t_1^2 t_K^4}{t_2^2(t_1^2 - t_K^2)^2} \\
R_3 &= \frac{1}{4} \ln \frac{(t_1^2 + t_K^2)\, t_2 + 2t_1 t_K^2}{(t_1^2 + t_K^2)\, t_2 - 2t_1 t_K^2} \\
R_4 &= \frac{1}{4} \ln \frac{(t_2^2 + t_K^2)\, t_1 + 2t_2 t_K^2}{(t_2^2 + t_K^2)\, t_1 - 2t_2 t_K^2} \\
R_5 &= \frac{1}{4} \ln \frac{t_1^2(t_2^2 + t_K^2)^2 - 4t_1^2 t_K^4}{t_1^2(t_2^2 - t_K^2)^2} \\
R_6 &= \frac{1}{4} \ln \frac{t_2^2 + 3t_K^2}{t_2^2 - t_K^2}
\end{aligned}
\tag{A.1b}
$$

where $t_\mu \equiv \tanh L_\mu$, $\mu = 1, 2$, and $t_K \equiv \tanh K$.

APPENDIX B

In this Appendix, we explicitly show the expression for the quenched free energy (3.2) corresponding to the double-delta probability distribution (3.1):

$$
\begin{aligned}
\overline{f_{\text{Kag}}} = \frac{1}{3} \Big\{ & \frac{p}{2}(-5 + p) \ln t_1 + \left(-2 + \frac{3}{2} p + \frac{p^2}{2}\right) \ln t_2 \\
& - \frac{3}{2} \ln(1 - t_K^2) + \frac{p}{2} \ln(1 - t_1^2) + \frac{1-p}{2} \ln(1 - t_2^2) \\
& + \frac{p}{2} \ln(t_1^2 - t_K^2) + \frac{1-p}{2} \ln(t_2^2 - t_K^2) + \frac{p^2}{4} \ln(1 + t_1^2 + 2t_K) \\
& + \frac{(1-p)^2}{4} \ln(1 + t_2^2 + 2t_K) + \frac{p^2}{4} \ln(1 + t_1^2 - 2t_K) \\
& + \frac{(1-p)^2}{4} \ln(1 + t_2^2 - 2t_K) + \frac{p(1-p)}{4} \ln[t_2(1 + t_1^2) - 2t_1 t_K] \\
& + \frac{p(1-p)}{4} \ln[t_2(1 + t_1^2) + 2t_1 t_K] \\
& + \frac{p(1-p)}{4} \ln[t_1(1 + t_2^2) - 2t_2 t_K]
\end{aligned}
$$

$$+\frac{p(1-p)}{4}\ln[t_1(1+t_2^2)+2t_2t_K]$$

$$+\frac{p^2}{4}\ln(t_1^2+t_K^2+2t_1^2t_K)$$

$$+\frac{(1-p)^2}{4}\ln(t_2^2+t_K^2+2t_2^2t_K)$$

$$+\frac{p^2}{4}\ln(t_1^2+t_K^2-2t_1^2t_K)$$

$$+\frac{(1-p)^2}{4}\ln(t_2^2+t_K^2-2t_2^2t_K)$$

$$+\frac{p(1-p)}{4}\ln(t_2^2+t_K^2+2t_1t_2t_K)$$

$$+\frac{p(1-p)}{4}\ln(t_1^2+t_K^2+2t_1t_2t_K)$$

$$+\frac{p(1-p)}{4}\ln(t_2^2+t_K^2-2t_1t_2t_K)$$

$$+\frac{p(1-p)}{4}\ln(t_1^2+t_K^2-2t_1t_2t_K)-\ln 2+\frac{2}{3}f_h \qquad \text{(B.1)}$$

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